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An Anisotropic Magnetized Viscous Fluid Cosmological Model in General Relativity

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We investigate the behavior of a magnetic field in a viscous fluid cosmological model where the free gravitational field is of Petrov type D and the coefficient of shear viscosity is proportional to the rate of expansion in the model. Also discussed are the behavior of the model when the magnetic field tends to zero and some other physical properties.

1. INTRODUCTION

Cosmological models which are anisotropic and homogeneous have a significant role in the description of the universe in the early stages of its evolution. A realistic treatment of the problem requires the consideration of a material distribution other than a perfect fluid. In the early stages of the universe, with radiation in the form of photons as well as neutrinos decoupled from matter, it behaved like a viscous fluid. It is also conjectured that there was a strong magnetic field contributing to the total energy of the system, and the coefficient of shear viscosity decreases as the universe expands. It is therefore reasonable to assume that the coefficient of shear viscosity is proportional to the rate of expansion.

Roy and Prakash (1976, 1977) obtained a viscous fluid cosmological model of plane symmetry. Bali (1985) obtained an expanding and shearing magnetoviscous fluid cosmological model in general relativity. Bali and Tyagi (1987) obtained a viscous fluid cosmological model of cylindrical symmetry in the presence of a magnetic field in which the coefficient of shear viscosity is assumed to be constant. In this paper, we obtain a magnetized viscous fluid cosmological model in which the coefficient of shear viscosity is proportional to the rate of expansion and the free gravitational field is that of Petrov type D. The distribution consists of an electrically neutral

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viscous fluid with an infinite electrical conductivity in the presence of a magnetic field. The various particular cases in which the magnetic field or viscosity or both tend to zero are also discussed.

The space-time is taken in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2\, dy^2 + C^2\, dz^2 \tag{1}$$

where A, B, and C are functions of t alone. The energy-momentum tensor is taken to be the sum of the energy-momentum tensors M_{ij} corresponding to a viscous fluid (Landau and Lifshitz, 1963) and E_{ij}, the electromagnetic field (Lichnerowicz, 1967), given by

$$\begin{aligned} M_i^j = (\varepsilon + p)v_i v^j + p g_i^j - (v_{i;}^j v_{;i}^j + v^j v^l v_{i;l} \\ + v_i v^l v_{;l}^j) - (\zeta - \tfrac{2}{3}\eta) v_{;l}^l (g_i^j + v_i v^j) \end{aligned} \tag{2}$$

and

$$E_i^j = \bar{\mu}\{|h|^2(v_i v^j + \tfrac{1}{2} g_i^j) - h_i h^j\} \tag{3}$$

In the above ε is the density, p is the pressure, η and ζ are the two coefficients of viscosity, and v^i is the flow vector satisfying the equation

$$g_{ij} v^i v^j = -1 \tag{4}$$

$\bar{\mu}$ being the magnetic permeability and h_i the magnetic flux vector defined by

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}}\, \varepsilon_{ijkl} F^{kl} v^j \tag{5}$$

where F_{kl} is the electromagnetic field tensor and ε_{ijkl} is the Levi-Civita tensor density. A semicolon stands for covariant differentiation. We take the incident magnetic field to be in the direction of the x axis, so that $h_1 \neq 0$, $h_2 = 0 = h_3 = h_4$. This leads to $F_{12} = 0 = F_{13}$ by virtue of equation (5). Also, $F_{14} = F_{24} = F_{34} = 0$ due to the assumption of infinite conductivity of the fluid. Hence, the only nonvanishing component of F_{ij} is F_{23}. The first set of Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \tag{6}$$

leads to $F_{23} = \text{const} = H$ (say). We also assume the coordinates to be comoving, so that $v^1 = 0 = v^2 = v^3$ and $v^4 = 1/A$.

The field equations

$$R_i^j - \tfrac{1}{2} R g_i^j + \Lambda g_i^j = -8\pi T_i^j \tag{7}$$

for the line element (1) are

$$\frac{1}{A^2}\left(-\frac{B_{44}}{B}-\frac{C_{44}}{C}-\frac{B_4C_4}{BC}+\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}\right)-\Lambda$$

$$=8\pi\left[p-\frac{2\eta A_4}{A^2}-\left(\zeta-\frac{2}{3}\eta\right)v^l_{;l}-\frac{H^2}{2\bar{\mu}B^2C^2}\right] \quad (8)$$

$$\frac{1}{A^2}\left(-\frac{C_{44}}{C}-\frac{A_{44}}{A}+\frac{A_4^2}{A^2}\right)-\Lambda$$

$$=8\pi\left[p-\frac{2\eta B_4}{AB}-\left(\zeta-\frac{2}{3}\eta\right)v^l_{;l}+\frac{H^2}{2\bar{\mu}B^2C^2}\right] \quad (9)$$

$$\frac{1}{A^2}\left(-\frac{B_{44}}{B}-\frac{A_{44}}{A}+\frac{A_4^2}{A^2}\right)-\Lambda$$

$$=8\pi\left[p-\frac{2\eta C_4}{AC}-\left(\zeta-\frac{2}{3}\eta\right)v^l_{;l}+\frac{H^2}{2\bar{\mu}B^2C^2}\right] \quad (10)$$

$$\frac{1}{A^2}\left(\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}+\frac{B_4C_4}{BC}\right)+\Lambda=8\pi\left(\varepsilon+\frac{H^2}{2\bar{\mu}B^2C^2}\right) \quad (11)$$

2. SOLUTION OF THE FIELD EQUATIONS

Equations (8)-(11) are four equations in five unknowns, A, B, C, ε, and p. For the complete determination of the set, we assume that the space-time is Petrov type D. This requires that

$$C^{12}_{12}=C^{13}_{13} \quad (12)$$

The condition is satisfied if $B=C$. However, we shall assume that A, B, and C are unequal due to the assumed anisotropy. From equations (7)-(9), we have

$$\left(\frac{A_4}{A}\right)_4+\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)-\frac{B_{44}}{B}-\frac{B_4C_4}{BC}$$

$$=16\pi\eta A\left(\frac{B_4}{B}-\frac{A_4}{A}\right)-\frac{8\pi H^2}{\bar{\mu}}\frac{A^2}{B^2C^2} \quad (13)$$

and

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}=16\pi\eta A\left(\frac{C_4}{C}-\frac{B_4}{B}\right) \quad (14)$$

Now, the condition $C^{12}_{12} = C^{13}_{13}$ leads to

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} + 2\frac{A_4}{A}\left(\frac{C_4}{C} - \frac{B_4}{B}\right) = 0 \tag{15}$$

From equations (14) and (15), we have

$$\left(\frac{C_4}{C} - \frac{B_4}{B}\right)\left(8\pi\eta A + \frac{A_4}{A}\right) = 0 \tag{16}$$

since $B \neq C$; hence, from equation (16), we have

$$8\pi\eta A + A_4/A = 0 \tag{17}$$

Here two cases arise: (i) $\eta = \text{const}$, (ii) $\eta/\theta = \text{const} = l$ (say). Considering case (i), the model has already been studied by Bali and Tyagi (1987). Hence we consider the case (ii), which leads to

$$\eta = \frac{l}{A}\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) \tag{18}$$

Equations (17) and (18) lead to

$$\frac{A_4}{A} = -\beta\left(\frac{B_4}{B} + \frac{C_4}{C}\right) \tag{19}$$

where

$$\beta = 8\pi l/(8\pi l + 1) \tag{20}$$

Putting $BC = \mu$ and $B/C = \nu$ in equations (14) and (19), we have

$$\frac{(\mu\nu_4/\nu)_4}{\mu\nu_4/\nu} = 2\frac{A_4}{A} \tag{21}$$

and

$$\frac{A_4}{A} = -\beta\frac{\mu_4}{\mu} \tag{22}$$

Equation (22) gives, after integration,

$$A = \gamma\mu^{-\beta} \tag{23}$$

where γ is a constant of integration; equation (21) leads to

$$\frac{\nu_4}{\nu} = k^2\mu^{-(2\beta+1)} \tag{24}$$

where $k^2 = m^2\gamma^2$ and m^2 is a constant of integration.

Now putting $BC = \mu$, $B/C = \nu$, and using equations (23) and (24) in equation (13), we have

$$
\begin{aligned}
&-\beta\left(\frac{\mu_{44}}{\mu}-\frac{\mu_4^2}{\mu^2}\right)-\beta\frac{\mu_4^2}{\mu^2}-\frac{1}{2}\left[\frac{\mu_{44}}{\mu}-\frac{\mu_4^2}{\mu^2}-k^2(2\beta+1)\mu^{-(2\beta+1)}\frac{\mu_4}{\mu}\right] \\
&-\frac{1}{4}\left(\frac{\mu_4^2}{\mu^2}+k^4\mu^{-2(2\beta+1)}+2k^2\mu^{-2(2\beta+1)}\frac{\mu_4}{\mu}\right) \\
&-\frac{1}{4}\left(\frac{\mu_4^2}{\mu^2}-k^4\mu^{-2(2\beta+1)}\right) \\
&=16\pi l(1-\beta)\frac{\mu_4}{\mu}\left[\left(\beta+\frac{1}{2}\right)\frac{\mu_4}{\mu}+\frac{k^2}{2}\mu^{-(2\beta+1)}\right]-\frac{8\pi H^2\gamma^2}{\bar{\mu}}\mu^{-2(\beta+1)} \qquad (25)
\end{aligned}
$$

which leads to

$$\mu\mu_{44}+2\beta\mu_4^2-\frac{L\gamma^2}{\beta+1/2}\mu^{-2\beta}=0 \qquad (26)$$

where $L = 8\pi H^2/\bar{\mu}$; equation (22) leads to

$$\mu_{44}+\frac{2\beta}{\mu}\mu_4^2=\frac{L\gamma^2}{\beta+1/2}\mu^{-2\beta-1} \qquad (27)$$

Inserting $\mu_4 = f(\mu)$ and $\mu_{44} = ff'$ in (27), we have

$$f=s^{1/2}\left[\frac{L\gamma^2}{S\beta(\beta+1/2)}\mu^{-2\beta}+\mu^{-4\beta}\right]^{1/2} \qquad (28)$$

where S is a constant of integration.

From equations (28) and (24), we have

$$\nu=N\left\{\mu^{-\beta}+\left[\mu^{-2\beta}+\frac{\gamma^2L}{S\beta(\beta+1/2)}\right]^{1/2}\right\}^{-k^2/\beta S^{1/2}} \qquad (29)$$

where N is a constant of integration.

Hence

$$A^2=\gamma^2\mu^{-2\beta} \qquad (30)$$

$$B^2=\mu N\left\{\mu^{-\beta}+\left[\mu^{-2\beta}+\frac{\gamma^2L}{S\beta(\beta+1/2)}\right]^{1/2}\right\}^{-k^2/\beta S^{1/2}} \qquad (31)$$

$$C^2=\frac{\mu}{N}\left\{\mu^{-\beta}+\left[\mu^{-2\beta}+\frac{\gamma^2L}{S\beta(\beta+1/2)}\right]^{1/2}\right\}^{k^2/\beta S^{1/2}} \qquad (32)$$

After suitable transformation of coordinates, the metric reduces to the form

$$ds^2 = \gamma^2 \bar{T}^{2\beta}\left[dX^2 - \frac{dT^2}{S[L\gamma^2 \bar{T}^{2\beta}/S\beta(\beta+1/2) + T^{-4\beta}]}\right]$$
$$+ 2^{L+k^2/\beta S^{1/2}} T\left\{T^{-\beta} + \left[T^{-2\beta} + \frac{\gamma^2 L}{S\beta(\beta+1/2)}\right]^{1/2}\right\}^{-k^2/\beta S^{1/2}} dY^2$$
$$+ \frac{T}{2^{L+k^2/\beta S^{1/2}}}\left\{T^{-\beta} + \left[T^{-2\beta} + \frac{\gamma^2 L}{S\beta(\beta+1/2)}\right]^{1/2}\right\}^{k^2/\beta S^{1/2}} dZ^2 \quad (33)$$

where $N = 2^{L+k^2/\beta S^{1/2}}$.

In the absence of a magnetic field, the metric (33) reduces to the form

$$ds^2 = \gamma^2 T^{-2\beta}\left(dX^2 - \frac{dT^2}{ST^{-4\beta}}\right) + T\cdot 2^{k^2/\beta S^{1/2}}(2T^{-\beta})^{-k^2/\beta S^{1/2}}\, dY^2$$
$$+ \frac{T}{2^{k^2/\beta S^{1/2}}}(2T^{-\beta})^{k^2/\beta S^{1/2}}\, dZ^2 \quad (34)$$

which in the absence of viscosity reduces to the form

$$ds^2 = (dX^2 - dT^2) + T^{1+k^2/S^{1/2}}\, dY^2 + T^{1-k^2/S^{1/2}}\, dZ^2 \quad (35)$$

3. SOME PHYSICAL AND GEOMETRICAL FEATURES

The pressure and density for the model (33) are given by

$$8\pi p = \frac{1}{4\gamma^2 T^{2(1-\beta)}}[(4\beta - k^4)ST^{-4\beta} + f^2] - \frac{16\pi\beta\eta f}{\gamma T^{1-\beta}}$$
$$+ 8\pi(\zeta - \tfrac{2}{3}\eta)\eta/l + \frac{8\pi H^2}{2\bar{\mu}T^2} - \Lambda \quad (36)$$

and

$$8\pi\varepsilon = \frac{1}{4\gamma^2 T^{2(1-\beta)}}[f^2(1-4\beta) - K^4 ST^{-4\beta}] - \frac{8\pi H^2}{2\bar{\mu}T^2} + \Lambda \quad (37)$$

where

$$f = \left[\frac{L\gamma^2 T^{-2\beta}}{\beta(\beta+1/2)} + ST^{-4\beta}\right]^{1/2} \quad (38)$$

The model (33) has to satisfy the reality conditions (Ellis, 1971)

(i) $\varepsilon + p > 0$ (39)

(ii) $\varepsilon + 3p > 0$ (40)

Condition (i) leads to

$$\frac{1}{16\pi\gamma^2 T^{2(1-\beta)}}[(2\beta-k^4)ST^{-4\beta}+f^2(1-2\beta)]$$
$$>\frac{2\beta f\eta}{\gamma T^{1-\beta}}-\left(\zeta-\frac{2}{3}\eta\right)\frac{\eta}{l}-\frac{H^2}{2\bar{\mu}T^2} \tag{41}$$

which is satisfied when $\beta<1$, $l>0$; condition (ii) leads to

$$\frac{1}{8\pi\gamma^2 T^{2(1-\beta)}}[(3\beta-k^4)ST^{-4\beta}+f^2(1-\beta)]$$
$$-\frac{6\beta f\eta}{\gamma T^{1-\beta}}+3\left(\zeta-\frac{2}{3}\eta\right)\frac{\eta}{l}+\frac{H^2}{\bar{\mu}T^2}>2\Lambda \tag{42}$$

which gives the condition on Λ.

Here

$$\eta=l\theta \tag{43}$$

$$\theta=\frac{f}{\gamma}\frac{1-\beta}{T^{1-\beta}} \tag{44}$$

The rotation ω is identically zero and the shear is given by

$$\sigma^2=\frac{f^2}{12\gamma^2 T^{2(1-\beta)}}[2\beta^2+8\beta-1]+\frac{s^{1/2}k^4}{4\gamma^2 T^{2(1+\beta)}} \tag{45}$$

The nonvanishing components of the conformal curvature tensor are given by

$$C^{12}_{12}=C^{13}_{13}=-\tfrac{1}{2}C^{23}_{23}$$
$$=\frac{1}{12\gamma^2 T^{2(1-\beta)}}[(\beta+1)(2\beta+1)f^2+(2\beta^2+\beta-k^4)ST^{-4\beta}] \tag{46}$$

For large values of T, the space-time is conformally flat. The model starts expanding at $T=0$ and goes on expanding indefinitely and the expansion stops for large values of T. From equations (37) and (44), we find that $\beta<1$, which leads to $\gamma>0$ and $l>0$. Since $\lim_{T\to\infty}(\sigma/\theta)\neq 0$, the model does not approach isotropy for large values of T.

In the absence of a magnetic field, the components of the conformal curvature tensor are given by

$$C^{12}_{12}=C^{13}_{13}=-\tfrac{1}{2}C^{23}_{23}=\frac{S}{12\gamma^2 T^{2(1+\beta)}}[4\beta^2+4\beta+1-k^4]$$
$$=\frac{S(1-k^4)}{12\gamma^2 T^2}\qquad\text{(absence of viscosity)}$$

The scalar of expansion in the absence of a magnetic field is given by

$$\theta = \frac{(1-\beta)\sqrt{s}}{\gamma T^{1+\beta}}$$

which tends to zero for large values of T. We notice that the isotropy is not attained for large values of T in the absence of a magnetic field also.

REFERENCES

Bali, R. (1985). *Astrophysics and Space Science*, **116**, 81.

Bali, R., and Tyagi, A. (1987). *Astrophysics and Space Science*, **138**, 173.

Ellis, G. F. R. (1971). In *General Relativity and Cosmology*, R. K. Sachs, ed., Academic Press, New York, p. 117.

Landau, L. D., and Lifshitz, E. M. (1963). *Fluid Mechanics*, Pergamon Press, Oxford, p. 505.

Lichnerowicz, A. (1967). *Relativistic Hydrodynamics and Magnetohydrodynamics*, New York, p. 63.

Roy, S. R., and Prakash, S. (1976). *Journal of Physics A: Mathematical General*, **9**, 261.

Roy, S. R., and Prakash, S. (1977). *Indian Journal of Pure and Applied Mathematics*, **8**, 723.